Construction of Induced Representations

$H \leq G$, $\qquad \varphi : H \longrightarrow GL(V)$

Construct: $\quad \psi : G \longrightarrow GL(W)$ such that

$$\chi_\psi = \mathrm{Ind}_H^G (\chi_\varphi)$$

$$W := \left\{ \omega : G \longrightarrow V \;\middle|\; \begin{array}{c} \omega(hx) = \varphi_h(\omega(x)) \\ \forall h \in H,\ x \in G \end{array} \right\}$$

$R = \{x_i\}$ = set of representatives of right H-cosets in G

Note: Values of ω are determined by $\{\omega(x_i)\}$

$$\dim W = (\dim V)\,[G:H]$$

Define $\psi : G \longrightarrow GL(W)$ by

$$\psi_g(\omega)(x) := \omega(xg)$$

$\omega \in W \qquad x \in G$

$\qquad\qquad\;\; g \in G$

Lemma: ψ is a representation.

$$W = \{ \omega : G \to V \mid \omega(hx) = \varphi_h(\omega(x)) \}$$

For each $Hx \subseteq G$,

$$W_{Hx} := \{ \omega \in W \mid \omega(y) = 0 \ , \ y \notin Hx \}$$

Lemma: W is a direct sum of these subspaces

$$W = W_{Hx_1} \oplus \cdots \oplus W_{Hx_m}, \qquad R = \{x_i\}$$

repr of right cosets

Lemma: $\psi_g(W_{Hx}) \subseteq W_{Hxg^{-1}}$

$\Rightarrow \quad \psi_g(W_{Hx}) \subseteq W_{Hx}$ iff $xgx^{-1} \in H$

Proof: $\omega \in W_{Hx}$, i.e. $\omega(y) = 0$ if $y \notin Hx$

let $\omega' = \psi_g(\omega)$

If $z \notin Hxg^{-1}$, $\Rightarrow$ $zg \notin Hx$

$$\omega'(z) = \psi_g(\omega)(z) = \omega(zg) = 0$$

$$W = \underline{W_{Hx_1}} \oplus \cdots \oplus W_{Hx_m}, \quad \underbrace{\psi_g(W_{H\underline{x}}) \subseteq W_{H\underline{x}g^{-1}}}$$

$$\psi_g \sim \begin{bmatrix} \boxed{\psi_g^{11}} & \psi_g^{12} & & \\ \psi_g^{21} & \boxed{\ddots} & & \\ & & \boxed{} & \\ & & & \boxed{\psi_g^{mm}} \end{bmatrix}$$

$$\psi_g^{ij} \in \mathrm{Hom}(W_{Hx_j}, W_{Hx_i})$$

$$\mathrm{Tr}\,\psi_g = \sum_{i=1}^{m} \mathrm{Tr}\,\psi_g^{ii} = \sum_{\substack{x_i \in R \\ x_i g x_i^{-1} \in H}}{}' \mathrm{Tr}(\psi_g|_{W_{Hx_i}})$$

Define: $E_x : W_{Hx} \xrightarrow{\simeq} V$

$$E_x(w) = w(x)$$

Lemma: $g, x \in G$, $xgx^{-1} \in H$, then

$$\underbrace{\psi_g|_{W_{Hx}}} = E_x^{-1} \circ \underbrace{\varphi_{xgx^{-1}}} \circ E_x$$

$$\mathrm{Tr}\,\psi_g = \sum_{\substack{x_i \in R \\ x_i g x_i^{-1} \in H}} \mathrm{Tr}(\varphi_{x_i g x_i^{-1}})$$

$$\Downarrow$$

$$\chi_\psi(g) = \sum_{\substack{x_i \in R \\ x_i g x_i^{-1} \in H}} \chi_\varphi(x_i g x_i^{-1})$$